Stability and convergence for a complete model of mass diffusion

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Abstract

We propose a fully discrete scheme for approximating a three-dimensional, strongly nonlinear model of mass diffusion, also called the complete Kazhikhov-Smagulov model. The scheme uses a C^0 finite element approximation for all unknowns (density, velocity and pressure), even though the density limit, solution of the continuous problem, belongs to H^2. A first order time discretization is used such that, at each time step, one only needs to solve two decoupled linear problems for the discrete density and the velocity-pressure, separately.

We extend to the complete model, some stability and convergence results already obtained by the last two authors for a simplified model where l^2-terms are not considered, l being the mass diffusion coefficient. Now, different arguments must be introduced, based mainly on an induction process with respect to the time step, obtaining at the same time the three main properties of the scheme: an approximate discrete maximum principle for the density, weak estimates for the velocity and strong ones for the density. Furthermore, the convergence towards a weak solution of the density-dependent Navier-Stokes problem is also obtained as $l \to 0$ (jointly with the space and time parameters).

Finally, some numerical computations prove the practical usefulness of the scheme.

Key words: three-dimensional Kazhikhov-Smagulov model, density-dependent Navier-Stokes problem, finite elements, stability, convergence.

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