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On numerical schemes for a Cahn Hilliard Diffuse Interface Model*

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Abstract

The Cahn-Hilliard model was originally introduced in [5] and describes the complicated phase separation and coarsening phenomena in the mixture of different fluids, solid or gas where only two different concentration phases can exist stably. Let $\Omega \in \mathbb{R}^d, d = 2, 3$ a bounded domain with boundary $\partial\Omega$. The model reads,

$$\begin{cases} \phi_t = \Delta w,\ w = -\Delta\phi + \dfrac{1}{\varepsilon^2}F'(\phi) & \text{in } \Omega \times (0,T) \\ \dfrac{\partial\phi}{\partial n} = 0,\ \dfrac{\partial w}{\partial n} = 0 & \text{in } \partial\Omega \end{cases}$$

where ϕ represents the phase field function, $F(\phi)$ is a double well potential and ε is a small parameter known as 'interaction length'. The following energy law holds

$$E(\phi) = \int_\Omega \left(\frac{1}{2}|\nabla\phi|^2 + F(\phi)\right) dx.$$

Numerical schemes to approximate the Cahn-Hilliard equation have been widely studied in recent times due to its connection with many physically motivated problems. In this work we propose two different ways to approximate the double-well potential term, driving to two new linear schemes. The first one is optimal from the numerical dissipation point of view meanwhile the second one allows us to design unconditionally stable linear schemes. We present first and second order in time linear schemes to approximate this problem, detailing their advantages over other linear schemes that have been previously introduced in the literature. Furthermore, we compare all the schemes through several computational experiments.

Key words: diffuse interface phase-field, Cahn-Hilliard, mixed finite element, long time stability.

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